

BIJECTIONS ON ROOTED TREES WITH FIXED SIZE OF MAXIMAL DECREASING SUBTREES

JANG SOO KIM

ABSTRACT. Seo and Shin showed that the number of rooted trees on $[n + 1]$ such that the maximal decreasing subtree with the same root has $k + 1$ vertices is equal to the number of functions $f : [n] \rightarrow [n]$ such that the image of f contains $[k]$. We give a bijective proof of this theorem.

1. INTRODUCTION

A *tree* on a finite set X is a connected acyclic graph with vertex set X . A *rooted tree* is a tree with a distinguished vertex called a *root*. It is well-known that the number of rooted trees on $[n] = \{1, 2, \dots, n\}$ is n^{n-1} , see [4, 5.3.2 Proposition].

Suppose T is a rooted tree with root r . For a vertex $v \neq r$ of T there is a unique path (u_1, u_2, \dots, u_i) from $r = u_1$ to $v = u_i$. Then u_{i-1} is called the *parent* of v , and v is called a *child* of u_{i-1} . For two vertices u and v , we say that u is a *descendant* of v if the unique path from r to u contains v . Note that every vertex is a descendant of itself. A *leaf* is a vertex with no children. A rooted tree is *decreasing* if every nonleaf is greater than its children. The *maximal decreasing subtree* of T , denoted $\text{MD}(T)$, is the maximal subtree such that it has the same root as T and it is decreasing. If the root of T has no smaller children, T is called *minimally rooted*.

The notion of maximal decreasing subtree was first appeared in [2] in order to prove the following theorem.

Theorem 1.1. [2, Theorem 2.1] *The number of rooted trees on $[n + 1]$ such that the root has ℓ smaller children equals $\binom{n}{\ell} n^{n-\ell}$.*

Recently, maximal decreasing subtrees reappeared in the study of a certain free Lie algebra over rooted trees by Bergeron and Livernet [1]. Seo and Shin [3] proved some enumeration properties of rooted trees with fixed size of maximal decreasing subtrees.

We denote by $\mathfrak{T}_{n,k}$ the set of rooted trees on $[n + 1]$ whose maximal decreasing subtrees have $k + 1$ vertices. Let $\mathfrak{F}_{n,k}$ denote the set of functions $f : [n] \rightarrow [n]$ such that $[k] \subset f([n])$, where $[0] = \emptyset$. Equivalently, $\mathfrak{F}_{n,k}$ is the set of words on $[n]$ of length n such that each of $1, 2, \dots, k$ appears at least once. Using the Prüfer code one can easily see that $\mathfrak{F}_{n,k}$ is in bijection with the set of rooted trees on $[n + 1]$ such that $n + 1$ is a leaf and $1, 2, \dots, k$ are nonleaves. Thus, we will consider $\mathfrak{F}_{n,k}$ as the set of such trees.

Seo and Shin [3] proved the following theorem.

Theorem 1.2. [3] *We have*

$$|\mathfrak{T}_{n,k}| = |\mathfrak{F}_{n,k}|.$$

In [3] they showed Theorem 1.2 by finding formulas for both sides and computing the formulas. In this paper we provide a bijective proof Theorem 1.2, which consists of several bijections between certain objects, see Theorem 1.3. In order to state the objects in Theorem 1.3 we need the following definitions.

An *ordered forest* on a finite set X is an ordered tuple of rooted trees whose vertex sets form a partition of X . We say that an ordered forest $(T_0, T_1, \dots, T_\ell)$ is *k-good* if it satisfies the following conditions:

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- (1) If $\ell = 0$, then T_0 has only one vertex v and we have $v \in [k]$.
- (2) If $\ell \geq 1$, then T_1, T_2, \dots, T_ℓ are minimally rooted, and the number of vertices of T_0, T_1, \dots, T_i contained in $[k]$ is at least $i + 1$ when $i = 0, 1, 2, \dots, \ell - 1$, and equal to ℓ when $i = \ell$.

We now state the main theorem of this paper.

Theorem 1.3. *The following sets have the same cardinality:*

- (1) the set $\mathfrak{T}_{n,k}$ of rooted trees on $[n+1]$ whose maximal decreasing subtrees have $k+1$ vertices,
- (2) the set $\mathfrak{A}_{n,k}$ of cycles of $k+1$ minimally rooted trees such that the vertex sets of the trees form a partition of $[n+1]$,
- (3) the set $\mathfrak{B}_{n,k}$ of ordered forests on $[n]$ such that the last k trees are minimally rooted,
- (4) the set $\mathfrak{C}_{n,k}$ of sequences of k -good ordered forests such that the vertex sets of the forests form a partition of $[n]$,
- (5) the set $\mathfrak{F}_{n,k}$ of rooted trees on $[n+1]$ such that $n+1$ is a leaf, and $1, 2, \dots, k$ are nonleaves.

In Section 2 we find bijections proving Theorem 1.3. The ideas in the bijections have some applications. In Section 3 we find a bijective proof of the following identity, which (finding a bijective proof) is stated as an open problem in [3]:

$$\sum_{k \geq 0} \frac{1}{k} |\mathfrak{T}_{n,k}| = n^n.$$

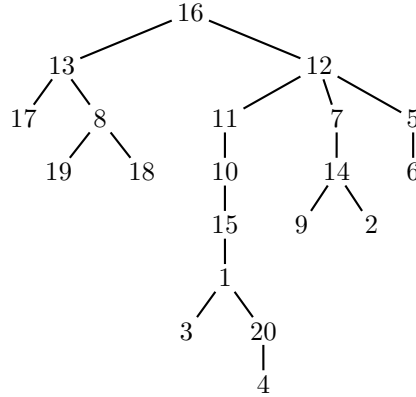
In Section 4, we give another bijective proof of Theorem 1.1.

From now on all trees in this paper are rooted trees.

2. BIJECTIONS

In this section we will find four bijections to prove Theorem 1.3. We assume that n and k are fixed nonnegative integers. We will write cycles using brackets to distinguish them from sequences. For instance, $[a_1, a_2, a_3]$ is a cycle and (a_1, a_2, a_3) is a sequence, thus $[a_1, a_2, a_3] = [a_2, a_3, a_1]$ and $(a_1, a_2, a_3) \neq (a_2, a_3, a_1)$. For a tree or a forest T , we denote by $V(T)$ the set of vertices in T .

2.1. A bijection $\alpha : \mathfrak{T}_{n,k} \rightarrow \mathfrak{A}_{n,k}$. We will explain the map α by an example. Let $T \in \mathfrak{T}_{19,7}$ be the following tree.



Then we can decompose T as follows:

$$(1) \quad T \Leftrightarrow \left(\begin{array}{c} \begin{array}{c} 16 \\ / \quad \backslash \\ 13 \quad 12 \\ / \quad \backslash \quad / \quad \backslash \\ 8 \quad 11 \quad 7 \quad 5 \\ | \quad | \\ 10 \end{array} \quad , \quad \begin{array}{c} 13 \\ | \\ 17 \end{array} \quad , \quad \begin{array}{c} 10 \\ | \\ 15 \\ | \\ 1 \\ / \quad \backslash \\ 3 \quad 20 \\ | \\ 4 \end{array} \quad , \quad \begin{array}{c} 8 \\ / \quad \backslash \\ 19 \quad 18 \end{array} \quad , \quad \begin{array}{c} 7 \\ | \\ 14 \\ / \quad \backslash \\ 9 \quad 2 \end{array} \quad , \quad \begin{array}{c} 5 \\ | \\ 6 \end{array} \right),$$

where the first tree is $\text{MD}(T)$, and the rest are the trees with more than one vertex in the forest obtained from T by removing the edges in $\text{MD}(T)$. We now construct a cycle C corresponding to

$\text{MD}(T)$ as follows. First, let C be the cycle containing only the maximal vertex m , which is the root of $\text{MD}(T)$. For each remaining vertex v , starting from the largest vertex to the smallest vertex, we insert v in C after the parent of v . In the current example, we get the cycle $[16, 12, 5, 7, 11, 10, 13, 8]$. It is easy to see that this process is invertible. In fact this is equivalent to the well-known algorithm called the depth-first search (preorder).

For each element v except the largest element in this cycle, if there is a tree with root v in (1) replace v with the tree. We then define $\alpha(T)$ to be the resulting cycle. It is not hard to see that α is a bijection. In the current example, we have

$$(2) \quad \alpha(T) = \left[\begin{array}{ccccccc} 16, & 12, & 5, & 7, & 11, & 10, & 13, \\ \begin{array}{c} | \\ 6 \end{array}, & \begin{array}{c} | \\ 14 \\ / \quad \backslash \\ 9 \quad 2 \end{array}, & & \begin{array}{c} | \\ 15 \\ | \\ 1 \\ / \quad \backslash \\ 3 \quad 20 \\ | \\ 4 \end{array}, & \begin{array}{c} | \\ 17 \end{array}, & \begin{array}{c} 19 \quad 18 \end{array} \end{array} \right] \in \mathfrak{B}_{19,7}.$$

2.2. A bijection $\beta : \mathfrak{A}_{n,k} \rightarrow \mathfrak{B}_{n,k}$. In order to define the map β we need two bijections ϕ and ρ in the following two lemmas. These bijections will also be used in other subsections.

Lemma 2.1. [2] *Let $A \subset [n]$. There is a bijection ϕ from the set of minimally rooted trees on A to the set of rooted trees on A such that $\max(A)$ is a leaf.*

Proof. We will briefly describe the bijection ϕ . See [2] for the details. Consider a minimally rooted tree T on A with root r . Let P be the subtree of T rooted at $\max(A)$ containing all descendants of $\max(A)$, and let Q be the tree obtained from T by deleting P (including $\max(A)$). We now consider the forest obtained from P by removing all edges of $\text{MD}(P)$. Suppose this forest has ℓ trees T_1, T_2, \dots, T_ℓ rooted at, respectively, r_1, r_2, \dots, r_ℓ . If $V(\text{MD}(P)) = \{u_1 < u_2 < \dots < u_t\}$ and $V(\text{MD}(P)) \setminus \{\max(A)\} \cup \{r\} = \{v_1 < v_2 < \dots < v_t\}$, let T' be the tree obtained from $\text{MD}(P)$ by replacing u_i with v_i for all i . Then $\phi(T)$ is the tree obtained from T' by attaching T_i at r_i for $i = 1, 2, \dots, \ell$ and attaching Q at r . \square

Lemma 2.2. *Let $A \subset [n]$. There is a bijection ρ from the set of rooted trees on A such that $\max(A)$ is a leaf to the set of ordered forests on $A \setminus \{\max(A)\}$.*

Proof. Suppose T is a rooted tree on A such that $\max(A)$ is a leaf. Let $r = r_1, r_2, \dots, r_{\ell+1} = \max(A)$ be the unique path from the root r of T to the leaf $\max(A)$. Let R_1, R_2, \dots, R_ℓ be the rooted trees with roots r_1, r_2, \dots, r_ℓ respectively in the forest obtained from T by removing the edges $r_1 r_2, r_2 r_3, \dots, r_\ell r_{\ell+1}$ and the vertex $r_{\ell+1} = \max(A)$. We define $\rho(T) = (R_1, R_2, \dots, R_\ell)$. It is easy to see that ρ is a desired bijection. \square

Let $[T_0, T_1, \dots, T_k] \in \mathfrak{A}_{n,k}$. Since $[T_0, T_1, \dots, T_k]$ is a cycle, we can assume that $n+1 \in T_0$. By Lemmas 2.1 and 2.2, the vertex $n+1$ in $\phi(T_0)$ is a leaf, and $\rho(\phi(T_0)) = (R_1, R_2, \dots, R_\ell)$ is an ordered forest on $V(T_0) \setminus \{n+1\}$. We define $\beta([T_0, T_1, \dots, T_k]) = (R_1, R_2, \dots, R_\ell, T_1, T_2, \dots, T_k)$. Since both ϕ and ρ are invertible, β is a bijection.

Example 1. Let \mathcal{F} be the cycle in (2). Then we can write \mathcal{F} as

$$\mathcal{F} = \left[\begin{array}{ccccccc} 10, & 13, & 8, & 16, & 12, & 5, & 7, \\ \begin{array}{c} | \\ 15 \\ | \\ 1 \\ / \quad \backslash \\ 3 \quad 20 \\ | \\ 4 \end{array}, & \begin{array}{c} | \\ 17 \end{array}, & \begin{array}{c} 19 \quad 18 \end{array}, & \begin{array}{c} | \\ 6 \end{array}, & \begin{array}{c} | \\ 14 \\ / \quad \backslash \\ 9 \quad 2 \end{array}, & & 11 \end{array} \right].$$

Then

$$(\rho \circ \phi) \left(\begin{array}{c} 10 \\ | \\ 15 \\ | \\ 1 \\ / \quad \backslash \\ 3 \quad 20 \\ | \\ 4 \end{array} \right) = \rho \left(\begin{array}{c} 10 \quad 4 \\ \backslash \quad / \\ 15 \\ | \\ 1 \\ / \quad \backslash \\ 3 \quad 20 \end{array} \right) = \left(\begin{array}{c} 10 \\ | \\ 4 \end{array}, \begin{array}{c} 15 \\ | \\ 3 \end{array}, \begin{array}{c} 1 \\ | \\ 3 \end{array} \right).$$

Thus

$$\beta(\mathcal{F}) = \left(\begin{array}{c} 10 \\ | \\ 4 \end{array}, \begin{array}{c} 15 \\ | \\ 3 \end{array}, \begin{array}{c} 1 \\ | \\ 3 \end{array}, \begin{array}{c} 13 \\ | \\ 17 \end{array}, \begin{array}{c} 8 \\ / \quad \backslash \\ 19 \quad 18 \end{array}, \begin{array}{c} 16 \\ | \\ 12 \end{array}, \begin{array}{c} 5 \\ | \\ 6 \end{array}, \begin{array}{c} 7 \\ | \\ 14 \\ / \quad \backslash \\ 9 \quad 2 \end{array}, \begin{array}{c} 11 \end{array} \right) \in \mathfrak{B}_{19,7}.$$

2.3. A bijection $\gamma : \mathfrak{B}_{n,k} \rightarrow \mathfrak{C}_{n,k}$. We call a vertex with label less than or equal to k a *special vertex*. For two ordered forests \mathcal{X} and \mathcal{Y} whose vertex sets are disjoint and contained in $[n]$, the pair $(\mathcal{X}, \mathcal{Y})$ is called a *balanced pair* if the trees in \mathcal{Y} are minimally rooted and the number of special vertices in \mathcal{X} and \mathcal{Y} is equal to the number of trees in \mathcal{Y} .

For two sets A and B , the *disjoint union* $A \uplus B$ is just the union of A and B . However, if we write $A \uplus B$, it is always assumed that $A \cap B = \emptyset$.

Lemma 2.3. *There is a bijection f from the set of balanced pairs $(\mathcal{X}, \mathcal{Y})$ to the set of pairs $(\mathcal{A}, (\mathcal{X}', \mathcal{Y}'))$ of a k -good ordered forest \mathcal{A} and a balanced pair $(\mathcal{X}', \mathcal{Y}')$ such that $V(\mathcal{X}) \uplus V(\mathcal{Y}) = V(\mathcal{A}) \uplus V(\mathcal{X}') \uplus V(\mathcal{Y}')$.*

Proof. Suppose $\mathcal{X} = (X_1, X_2, \dots, X_s)$ and $\mathcal{Y} = (Y_1, Y_2, \dots, Y_t)$. We define $f(\mathcal{X}, \mathcal{Y}) = (\mathcal{A}, (\mathcal{X}', \mathcal{Y}'))$ as follows.

Case 1: If $s \geq 1$, and X_1 does not contain a special vertex, we define $\mathcal{A} = (X_1)$, $\mathcal{X}' = (X_2, \dots, X_s)$, and $\mathcal{Y}' = \mathcal{Y}$.

Case 2: If $s \geq 1$, and X_1 contains at least one special vertex, there is a unique integer $1 \leq j \leq t$ such that $(X_1, Y_1, Y_2, \dots, Y_j)$ is a k -good ordered forest. Then we define $\mathcal{A} = (X_1, Y_1, Y_2, \dots, Y_j)$, $\mathcal{X}' = (X_2, X_3, \dots, X_s)$, and $\mathcal{Y}' = (Y_{j+1}, Y_{j+2}, \dots, Y_t)$. Since \mathcal{A} is a k -good ordered forest, there are j special vertices in $X_1, Y_1, Y_2, \dots, Y_j$. This implies that $(\mathcal{X}', \mathcal{Y}')$ is also a balanced pair.

Case 3: If $s = 0$, then $\mathcal{X} = \emptyset$ and there are t special vertices in Y_1, Y_2, \dots, Y_t . Let $U = V(Y_1) \uplus \dots \uplus V(Y_s)$ and let $m = \max(U)$. Suppose Y_i contains m . We apply the map ϕ in Lemma 2.1 to Y_i . Then $\phi(Y_i)$ is a rooted tree such that m is a leaf. If we apply the map ρ in Lemma 2.2 to $\phi(Y_i)$, we get an ordered forest $\rho(\phi(Y_i)) = (T_1, T_2, \dots, T_\ell)$ on $V(Y_i) \setminus \{m\}$. Let $\overline{\mathcal{X}} = (T_1, T_2, \dots, T_\ell)$ and $\overline{\mathcal{Y}} = (Y_1, Y_2, \dots, \widehat{Y}_i, \dots, Y_t)$. Note that the set of vertices in $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$ is $U \setminus \{m\}$. Let $s_1 < s_2 < \dots < s_t$ be the special vertices in U . Suppose $U \setminus \{m\} = \{v_1 < v_2 < \dots < v_p\}$ and $U \setminus \{s_i\} = \{u_1 < u_2 < \dots < u_p\}$. Then we define \mathcal{X}' (resp. \mathcal{Y}') to be the ordered forest obtained from $\overline{\mathcal{X}}$ (resp. $\overline{\mathcal{Y}}$) by replacing v_j with u_j for all j . We define \mathcal{A} to be the rooted tree with only one vertex s_i . It is clear from the construction that \mathcal{A} is a k -good ordered forest and $(\mathcal{X}', \mathcal{Y}')$ is a balanced pair.

In all cases, we clearly have $V(\mathcal{X}) \uplus V(\mathcal{Y}) = V(\mathcal{A}) \uplus V(\mathcal{X}') \uplus V(\mathcal{Y}')$. We now show that f is invertible by constructing the inverse map $g = f^{-1}$. Suppose $\mathcal{A} = (A_1, \dots, A_r)$, $\mathcal{X}' = (X_1, \dots, X_s)$, and $\mathcal{Y}' = (Y_1, \dots, Y_t)$, where \mathcal{A} is a k -good forest and $(\mathcal{X}', \mathcal{Y}')$ is a balanced pair. Then we define $g(\mathcal{A}, (\mathcal{X}', \mathcal{Y}')) = (\mathcal{X}, \mathcal{Y})$ as follows.

Case 1: If $r = 1$, and A_1 does not have a special vertex, we define $\mathcal{X} = (A_1, X_1, \dots, X_s)$ and $\mathcal{Y} = \mathcal{Y}'$.

Case 2: If $r \geq 2$, we define $\mathcal{X} = (A_1, X_1, \dots, X_s)$ and $\mathcal{Y} = (A_2, \dots, A_r, Y_1, \dots, Y_t)$.

Case 3: If $r = 1$, and A_1 has a special vertex, then by definition of k -good forests, A_1 has only one vertex. Let U be the set of vertices in \mathcal{A} , \mathcal{X}' , and \mathcal{Y}' , and $m = \max(U)$. Suppose $s_1 < \dots < s_{t+1}$ are the $t+1$ special vertices in U , and the unique vertex in A_1 is s_j . Let $U \setminus \{m\} = \{v_1 < v_2 < \dots < v_p\}$ and $U \setminus \{s_j\} = \{u_1 < u_2 < \dots < u_p\}$. Then we define $\overline{\mathcal{X}} = (T_1, \dots, T_r)$ and $\overline{\mathcal{Y}} = (R_1, \dots, R_s)$ to be the ordered forests obtained from \mathcal{X}' and \mathcal{Y}' by

replacing u_i with v_i for all i . Then the set of vertices in \mathcal{X}' and \mathcal{Y}' is now $U \setminus \{m\}$. Thus we can construct the tree $T = \rho^{-1}(\mathcal{X}')$ with maximum label m , and $R = \phi^{-1}(T)$ is a minimally rooted tree. We define $\mathcal{X} = \emptyset$ and $\mathcal{Y} = (R_1, \dots, R_{i-1}, R, R_i, \dots, R_s)$.

It is easy to see that g is the inverse map of f . \square

Now we are ready to define the map γ . Suppose $(T_1, T_2, \dots, T_\ell, T_{\ell+1}, T_{\ell+2}, \dots, T_{\ell+k}) \in \mathfrak{B}_{n,k}$. Let $\mathcal{X} = (T_1, T_2, \dots, T_\ell)$ and $\mathcal{Y} = (T_{\ell+1}, T_{\ell+2}, \dots, T_{\ell+k})$. Then $(\mathcal{X}, \mathcal{Y})$ is a balanced pair. We define $(\mathcal{X}_0, \mathcal{Y}_0), (\mathcal{X}_1, \mathcal{Y}_1), \dots$, and $\mathcal{A}_1, \mathcal{A}_2, \dots$, as follows. Let $(\mathcal{X}_0, \mathcal{Y}_0) = (\mathcal{X}, \mathcal{Y})$. For $i \geq 0$, if $(\mathcal{X}_i, \mathcal{Y}_i) \neq (\emptyset, \emptyset)$, we define $\mathcal{A}_{i+1}, \mathcal{X}_{i+1}, \mathcal{Y}_{i+1}$ by $f(\mathcal{X}_i, \mathcal{Y}_i) = (\mathcal{A}_{i+1}, (\mathcal{X}_{i+1}, \mathcal{Y}_{i+1}))$. Let p be the smallest integer such that $\mathcal{X}_p = \mathcal{Y}_p = \emptyset$. Then we define $\gamma(\mathcal{X}, \mathcal{Y})$ to be $(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p) \in \mathfrak{C}_{n,k}$. Since f is invertible, γ is a bijection.

Example 2. Let

$$\mathcal{F} = \left(\begin{array}{c} 10 \\ | \\ 4 \end{array}, 15, \begin{array}{c} 1 \\ | \\ 3 \end{array}, \begin{array}{c} 13 \\ | \\ 17 \end{array}, \begin{array}{cc} 8 & \\ / & \backslash \\ 19 & 18 \end{array}, 16, 12, \begin{array}{c} 5 \\ | \\ 6 \end{array}, \begin{array}{c} 7 \\ | \\ 14 \\ / \backslash \\ 9 \quad 2 \end{array}, 11 \right) \in \mathfrak{B}_{19,7}.$$

Note that special vertices are less than or equal to 7. Then

$$\begin{aligned} \mathcal{X} = \mathcal{X}_0 &= \left(\begin{array}{c} 10 \\ | \\ 4 \end{array}, 15, \begin{array}{c} 1 \\ | \\ 3 \end{array} \right), \quad \mathcal{Y} = \mathcal{Y}_0 = \left(\begin{array}{c} 13 \\ | \\ 17 \end{array}, \begin{array}{cc} 8 & \\ / & \backslash \\ 19 & 18 \end{array}, 16, 12, \begin{array}{c} 5 \\ | \\ 6 \end{array}, \begin{array}{c} 7 \\ | \\ 14 \\ / \backslash \\ 9 \quad 2 \end{array}, 11 \right), \\ \mathcal{A}_1 &= \left(\begin{array}{c} 10 \\ | \\ 4 \end{array}, \begin{array}{c} 13 \\ | \\ 17 \end{array} \right), \quad \mathcal{X}_1 = \left(15, \begin{array}{c} 1 \\ | \\ 3 \end{array} \right), \quad \mathcal{Y}_1 = \left(\begin{array}{cc} 8 & \\ / & \backslash \\ 19 & 18 \end{array}, 16, 12, \begin{array}{c} 5 \\ | \\ 6 \end{array}, \begin{array}{c} 7 \\ | \\ 14 \\ / \backslash \\ 9 \quad 2 \end{array}, 11 \right), \\ \mathcal{A}_2 &= (15), \quad \mathcal{X}_2 = \left(\begin{array}{c} 1 \\ | \\ 3 \end{array} \right), \quad \mathcal{Y}_2 = \left(\begin{array}{cc} 8 & \\ / & \backslash \\ 19 & 18 \end{array}, 16, 12, \begin{array}{c} 5 \\ | \\ 6 \end{array}, \begin{array}{c} 7 \\ | \\ 14 \\ / \backslash \\ 9 \quad 2 \end{array}, 11 \right), \\ \mathcal{A}_3 &= \left(\begin{array}{c} 1 \\ | \\ 3 \end{array}, \begin{array}{cc} 8 & \\ / & \backslash \\ 19 & 18 \end{array}, 16 \right), \quad \mathcal{X}_3 = \emptyset, \quad \mathcal{Y}_3 = \left(\begin{array}{c} 12, 5 \\ | \quad | \\ 6 \end{array}, \begin{array}{c} 7 \\ | \\ 14 \\ / \backslash \\ 9 \quad 2 \end{array}, 11 \right). \end{aligned}$$

In \mathcal{Y}_3 the largest vertex 14 is in the third tree. Using ϕ and ρ we get

$$(\rho \circ \phi) \left(\begin{array}{c} 7 \\ | \\ 14 \\ / \backslash \\ 9 \quad 2 \end{array} \right) = \rho \left(\begin{array}{cc} 9 & \\ / & \backslash \\ 7 & 2 \\ | \\ 14 \end{array} \right) = \left(\begin{array}{c} 9 \\ | \\ 2 \end{array}, 7 \right).$$

Thus

$$\overline{\mathcal{X}}_3 = \left(\begin{array}{c} 9 \\ | \\ 2 \end{array}, 7 \right), \quad \overline{\mathcal{Y}}_3 = \left(12, \begin{array}{c} 5 \\ | \\ 6 \end{array}, 11 \right).$$

Since \mathcal{X}_3 and \mathcal{Y}_3 have 4 special vertices 2, 5, 6, 7, and 6 is the third smallest special vertex, we replace the vertices in $U \setminus \{14\}$ with those in $U \setminus \{6\}$. Since

$$\begin{aligned} U \setminus \{14\} &= \{ 2, 5, 6, 7, 9, 11, 12 \}, \\ U \setminus \{6\} &= \{ 2, 5, 7, 9, 11, 12, 14 \}, \end{aligned}$$

we get

$$\mathcal{A}_4 = (6), \quad \mathcal{X}_4 = \overline{\mathcal{X}}'_3 = \left(\begin{array}{c} 11 \\ | \\ 2 \end{array}, 9 \right), \quad \mathcal{Y}_4 = \overline{\mathcal{Y}}'_3 = \left(14, \begin{array}{c} 5 \\ | \\ 7 \end{array}, 12 \right),$$

$$\mathcal{A}_5 = \left(\begin{array}{c} 11 \\ | \\ 2 \end{array}, \quad 14 \right), \quad \mathcal{X}_5 = (9), \quad \mathcal{Y}_5 = \left(\begin{array}{c} 5 \\ | \\ 7 \end{array}, \quad 12 \right),$$

$$\mathcal{A}_6 = (9), \quad \mathcal{X}_6 = \emptyset, \quad \mathcal{Y}_6 = \left(\begin{array}{c} 5 \\ | \\ 7 \end{array}, \quad 12 \right).$$

In \mathcal{Y}_6 , the largest vertex 12 is in the second tree.

$$(\rho \circ \phi)(12) = \rho(12) = \emptyset.$$

Thus

$$\overline{\mathcal{X}}_6 = \emptyset, \quad \overline{\mathcal{Y}}_6 = \left(\begin{array}{c} 5 \\ | \\ 7 \end{array} \right).$$

If we replace the labels in $\{5, 7\}$ with $\{5, 12\}$ we get

$$\mathcal{A}_7 = (7), \quad \mathcal{X}_7 = \emptyset, \quad \mathcal{Y}_7 = \left(\begin{array}{c} 5 \\ | \\ 12 \end{array} \right).$$

Since

$$(\rho \circ \phi) \left(\begin{array}{c} 5 \\ | \\ 12 \end{array} \right) = \rho \left(\begin{array}{c} 5 \\ | \\ 12 \end{array} \right) = 5,$$

we have $\overline{\mathcal{X}}_7 = (5)$ and $\overline{\mathcal{Y}}_6 = \emptyset$. Replacing 5 with 12 we get

$$\mathcal{A}_8 = (5), \quad \mathcal{X}_8 = (12), \quad \mathcal{Y}_8 = \emptyset.$$

Finally we get

$$\mathcal{A}_9 = (12), \quad \mathcal{X}_9 = \emptyset, \quad \mathcal{Y}_9 = \emptyset.$$

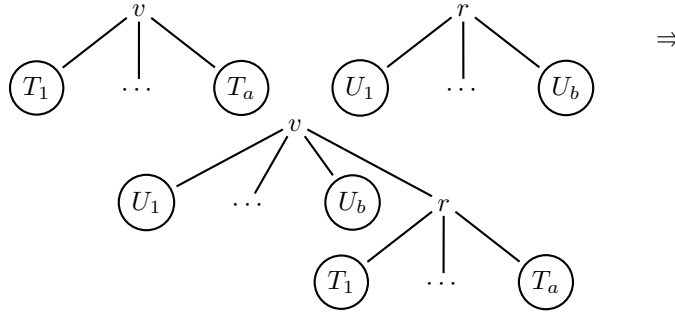
Thus

$$\gamma(\mathcal{F}) = \left(\left(\begin{array}{c} 10 \\ | \\ 4 \end{array}, \quad \begin{array}{c} 13 \\ | \\ 17 \end{array} \right), (15), \left(\begin{array}{c} 1 \\ | \\ 3 \end{array}, \quad \begin{array}{c} 8 \\ \swarrow \quad \searrow \\ 19 \quad 18 \end{array}, \quad 16 \right), (6), \left(\begin{array}{c} 11 \\ | \\ 2 \end{array}, \quad 14 \right), (9), (7), (5), (12) \right) \in \mathfrak{C}_{19,7}.$$

2.4. A bijection $\zeta : \mathfrak{C}_{n,k} \rightarrow \mathfrak{F}_{n,k}$. Recall that a special vertex is a vertex with label at most k .

Lemma 2.4. *For a fixed set $A \subset [n]$ with $|A| \geq 2$, there is a bijection ψ from the set of k -good ordered forests on A to the set of rooted trees on A whose special vertices are nonleaves.*

Proof. Suppose $\mathcal{F} = (A_1, A_2, \dots, A_p)$ is a k -good ordered forest. We first set all special vertices in \mathcal{F} active. Find the smallest vertex v among the active vertices with minimal distance from the root in A_1 . Then exchange the subtrees attached to v and those attached to the root r of A_2 , and then attach the resulting tree rooted at r to v as shown below.



We then make v inactive. Note that v is a nonleaf after this procedure. We do the same thing with the resulting tree and A_3 , and proceed until there are no active special vertices. Since (A_1, A_2, \dots, A_p) is k -good, we can eventually combine all of A_1, A_2, \dots, A_p into a single rooted tree in which the special vertices are nonleaves. We define $\psi(\mathcal{F})$ to be the resulting tree. It is straightforward to check that ψ is invertible. \square

Let $(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_h) \in \mathfrak{C}_{n,k}$. For each k -good forest \mathcal{F}_i we define $T_i = \psi(\mathcal{F}_i)$ if \mathcal{F}_i has at least 2 vertices, and $T_i = X$ if $\mathcal{F}_i = (X)$ and X has only one vertex. Then we define $\zeta(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_h) = \rho^{-1}(T_1, \dots, T_h)$. Since ρ^{-1} and ψ are invertible, ζ is a bijection.

Example 3. Let $(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_h)$ be the following:

$$\left(\left(\begin{array}{c} 10 \\ | \\ 4 \end{array}, \begin{array}{c} 13 \\ | \\ 17 \end{array} \right), (15), \left(\begin{array}{c} 1 \\ | \\ 3 \end{array}, \begin{array}{c} 8 \\ / \quad \backslash \\ 19 \quad 18 \end{array}, 16 \right), (6), \left(\begin{array}{c} 11 \\ | \\ 2 \end{array}, 14 \right), (9), (7), (5), (12) \right) \in \mathfrak{C}_{19,7}.$$

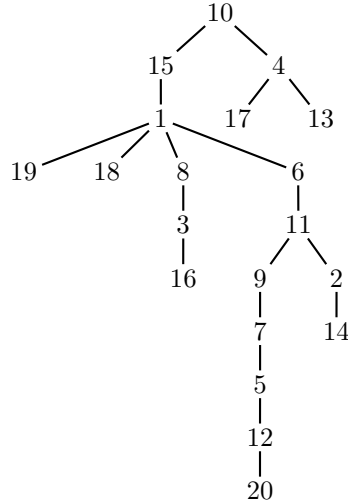
Then the map ψ sends

$$\begin{aligned} \left(\begin{array}{c} 10 \\ | \\ 4 \end{array}, \begin{array}{c} 13 \\ | \\ 17 \end{array} \right) &\mapsto \begin{array}{c} 10 \\ | \\ 4 \\ / \quad \backslash \\ 17 \quad 13 \end{array}, \\ \left(\begin{array}{c} 1 \\ | \\ 3 \end{array}, \begin{array}{c} 8 \\ / \quad \backslash \\ 19 \quad 18 \end{array}, 16 \right) &\mapsto \left(\begin{array}{c} 1 \\ / \quad | \quad \backslash \\ 19 \quad 18 \quad 8 \\ | \\ 3 \end{array}, 16 \right) \mapsto \begin{array}{c} 1 \\ / \quad | \quad \backslash \\ 19 \quad 18 \quad 8 \\ | \quad | \quad | \\ 3 \quad 16 \end{array}, \\ \left(\begin{array}{c} 11 \\ | \\ 2 \end{array}, 14 \right) &\mapsto \begin{array}{c} 11 \\ | \\ 2 \\ | \\ 14 \end{array}. \end{aligned}$$

Thus we obtain (T_1, \dots, T_h) :

$$\left(\begin{array}{c} 10 \\ | \\ 4 \\ / \quad \backslash \\ 17 \quad 13 \end{array}, 15, \begin{array}{c} 1 \\ / \quad | \quad \backslash \\ 19 \quad 18 \quad 8 \\ | \\ 3 \\ | \\ 16 \end{array}, 6, \begin{array}{c} 11 \\ | \\ 2 \\ | \\ 14 \end{array}, 9, 7, 5, 12 \right)$$

If we add the vertex $n + 1$, we obtain $\zeta(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_h)$:



3. SOME PROPERTIES OF $|\mathfrak{T}_{n,k}|$

We denote the cardinality of $|\mathfrak{T}_{n,k}|$ by $a_{n,k}$. In [3] Seo and Shin proved the following.

Theorem 3.1. [3] *We have*

$$(3) \quad \sum_{k \geq 0} \binom{m+k}{k} a_{n,k} = (n+m+1)^n,$$

$$(4) \quad \sum_{k \geq 0} \frac{1}{k} a_{n,k} = n^n.$$

We give another proof using generating functions.

Proof. By Theorem 1.2, $a_{n,k}$ equals the number of words of length n on $[n]$ with at least one i for all $i \in [k]$. Thus

$$a_{n,k} = \left[\frac{x^n}{n!} \right] (e^x - 1)^k e^{(n-k)x} = \left[\frac{x^n}{n!} \right] (1 - e^{-x})^k e^{nx},$$

where $[y^n] f(y)$ denotes the coefficient of y^n in $f(y)$. Then we have

$$\begin{aligned} \sum_{k \geq 0} \binom{m+k}{k} a_{n,k} &= \left[\frac{x^n}{n!} \right] e^{nx} \sum_{k \geq 0} \binom{m+k}{k} (1 - e^{-x})^k \\ &= \left[\frac{x^n}{n!} \right] e^{nx} \frac{1}{(1 - (1 - e^{-x}))^{m+1}} \\ &= \left[\frac{x^n}{n!} \right] e^{(n+m+1)x} = (n+m+1)^n, \end{aligned}$$

where the following binomial theorem [5, (1.20)] is used:

$$\frac{1}{(1-x)^n} = \sum_{k \geq 0} \binom{n+k-1}{k} x^k.$$

The second identity is proved similarly:

$$\begin{aligned} \sum_{k \geq 0} \frac{1}{k} a_{n,k} &= \left[\frac{x^n}{n!} \right] e^{nx} \sum_{k \geq 0} \frac{(1 - e^{-x})^k}{k} \\ &= \left[\frac{x^n}{n!} \right] e^{nx} \ln \frac{1}{1 - (1 - e^{-x})} \\ &= \left[\frac{x^n}{n!} \right] x e^{nx} = n! [x^{n-1}] e^{nx} = n! \frac{n^{n-1}}{(n-1)!} = n^n. \end{aligned}$$

□

In [3] they asked for a bijective proof of (4). We can prove it bijectively using our bijections as follows.

Bijective proof of (4). By Theorem 1.3, $a_{n,k}$ is also equal to $|\mathfrak{B}_{n,k}|$, the number of ordered forests $(T_1, T_2, \dots, T_\ell, T_{\ell+1}, \dots, T_{\ell+k})$ on $[n]$ such that $T_{\ell+1}, \dots, T_{\ell+k}$ are minimally rooted. Thus $a_{n,k}/k$ is equal to the number of pairs (\mathcal{F}, C) of an ordered forest $\mathcal{F} = (T_1, T_2, \dots, T_\ell)$ and a cycle $C = [T_{\ell+1}, \dots, T_{\ell+k}]$ of k minimally rooted trees such that the vertex sets of $T_1, \dots, T_{\ell+k}$ form a partition of $[n]$. Then, by Theorem 1.3, the number of cycles of k minimally rooted trees whose vertex sets form a subset A of $[n]$ is equal to the set of rooted trees T on A with $|\text{MD}(T)| = k$. Thus $a_{n,k}/k$ is equal to the number of ordered forests $(T_1, T_2, \dots, T_\ell, T)$ on $[n]$ with $|\text{MD}(T)| = k$. The sum of $a_{n,k}/k$ for all k is equal to the number of ordered forests on $[n]$. Suppose $(T_1, T_2, \dots, T_\ell)$ is an ordered forest on $[n]$ and r_i is the root of T_i for $i \in [\ell]$. By adding the edges $r_1 r_2, r_2 r_3, \dots, r_{\ell-1} r_\ell$, we get a rooted tree, say H . If we know the root r_ℓ of the last tree, then we can recover the ordered forest $(T_1, T_2, \dots, T_\ell)$ from H . Thus there is a bijection between the set of ordered forests on $[n]$ and the set of rooted trees on $[n]$ with a choice of r_ℓ . The latter set has cardinality n^n by Prüfer code. This proves (4). □

4. ANOTHER PROOF OF THEOREM 1.1

Using Prüfer code one can easily see that $\binom{n}{\ell}n^{n-\ell}$ is equal to the number of rooted trees on $\{0, 1, 2, \dots, n+1\}$ such that 0 is the root with $\ell+1$ children and $n+1$ is a leaf. By deleting the root 0, such a tree is identified with a forest on $[n+1]$ with $\ell+1$ rooted trees such that $n+1$ is a leaf. Thus by replacing $n+1$ with n , we can rewrite Theorem 1.1 as follows.

Theorem 4.1. [2] *The number of rooted trees of $[n]$ such that the root has ℓ smaller children equals the number of forests on $[n]$ with $\ell+1$ trees such that n is a leaf.*

Proof. Let T be a rooted trees of $[n]$ such that the root has ℓ smaller children. We will construct a forest corresponding to T . Recall the bijection $\alpha : \mathfrak{T}_{n,k} \rightarrow \mathfrak{A}_{n,k}$. Suppose $T \in \mathfrak{T}_{n-1,k}$, $\alpha(T) = [T_0, T_1, \dots, T_k]$, r_i is the root of T_i for $i = 0, 1, 2, \dots, k$. By shifting cyclically we can assume that r_0 is the largest root. Note that, by the construction of α , T is rooted at r_0 . Also, from the construction of α , it is easy to see that the smaller children of the root r_0 in T are exactly the left-to-right maxima of r_1, r_2, \dots, r_k . Suppose $r_{i_1} < r_{i_2} < \dots < r_{i_\ell}$ are the smaller children of r_0 in T . Then $1 = i_1 < i_2 < \dots < i_\ell \leq k$. Suppose n is contained in T_j . Let T'_1, T'_2, \dots, T'_k be the arrangement of the trees $T_0, T_1, \dots, \widehat{T_j}, \dots, T_k$ such that the word $r'_1 r'_2 \dots r'_k$ of the roots of T'_1, \dots, T'_k are order-isomorphic to the word $r_1 r_2 \dots r_k$. Notice that $r'_{i_1}, r'_{i_2}, \dots, r'_{i_\ell}$ are the left-to-right maxima of $r'_1 r'_2 \dots r'_k$. Thus the following map is invertible:

$$(5) \quad (T'_1, T'_2, \dots, T'_k) \mapsto \{[T'_{i_1}, \dots, T'_{i_2-1}], [T'_{i_2}, \dots, T'_{i_3-1}], \dots, [T'_{i_\ell}, \dots, T'_k]\}.$$

Now we apply the inverse map α^{-1} of the bijection α to each cycle in (5). Then we get a set of rooted trees. Together with T_j , we obtain a forest on $[n]$. Since T_j is the tree containing n , we can recover the original tree T from the forest. This gives a bijection between the two sets in the theorem. \square

The proof of Theorem 4.1, in fact, gives a generalization as follows.

Corollary 4.2. *Let $A(n, \ell, k)$ denote the number of rooted trees of $[n]$ such that the root has ℓ smaller children and the minimal decreasing subtree has $k+1$ vertices. Let $B(n, \ell, k)$ denote the number of forests on $[n]$ with $\ell+1$ trees such that n is a leaf, and the sum of $|\text{MD}(T)|$ for all trees T in the forest except the one containing n is equal to k . Then*

$$A(n, \ell, k) = B(n, \ell, k).$$

Proof. This can be checked by the following observation. Consider a cycle C in (5), and $T = \alpha^{-1}(C)$. Then $|\text{MD}(T)|$ is the number of trees in C . Thus the sum of $|\text{MD}(T)|$ for all cycles C in (5) is k . \square

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- E-mail address:* kimjs@math.umn.edu